Ryerson University

Department of Electrical and Computer Engineering

ELE401: Field Theory

Term Test #2

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Sections: 01 - 08
Time: ~ 1.75 hour

Instructions:

- **Closed book test.** Aids such as vector operators and integral tables will be provided if needed. **No questions are to be asked during the test.** If some aspect of a question is unclear make suitable assumptions -- write down the assumptions -- and continue solving the problem. **Calculators will not be required and are not permitted.**
- **Answer all three (3) questions**. Weight of each question is indicated. Part marks will be given. ***Note*** that there are **60 marks** available in total, but only **40 marks** are needed for a complete paper; i.e. **20 bonus marks** are possible.
- Clearly show all work in the space provided. **If a *"Right"* answer appears without any indication on how it was derived that answer will be considered to be *"Wrong"*.** Use the back sides of pages if more working space is needed. ***Underline*** or **box** your simplified answers and show proper **units.**

Questions	*Available Marks*	*Your Marks*
Question 1	20 marks	
Question 2	20 marks	
Question 3	20 marks	
Total Marks	*40 (60 with bonus)*	

Your Name (Please Print)	SOLUTIONS
Your Student Number	
Your Section Number	Circle your section number: 01, 02, 03, 04, 05, 06, 07, 08

ELE401: Field Theory: *Term Test #2* *Version "B"*

March 29, 2003

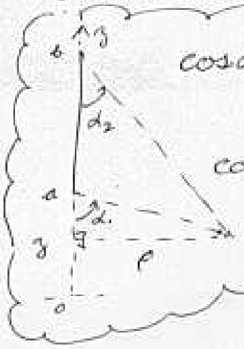
Problem 1: **Four Short Questions: [20 marks]**

(a) Magnetic Vector Field H Due To A Finite Line Of Current: [6 marks]

Use the expression for the **H** - field due to a short line of current "I" located on the z- axis, i.e.

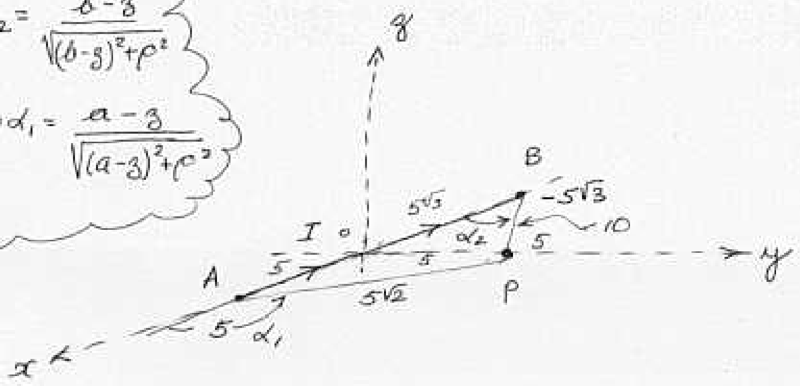
$$\boldsymbol{H} = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)\boldsymbol{a}_\phi \ [A/m]$$

to solve for the **H** - field caused by a short line of current I [A] flowing from point A(5, 0, 0) to point B(- 5 √3, 0, 0) as sensed at point P(0, 5, 0).



$$\cos\alpha_2 = \frac{b-z}{\sqrt{(b-z)^2+\rho^2}}$$

$$\cos\alpha_1 = \frac{a-z}{\sqrt{(a-z)^2+\rho^2}}$$



$$\therefore \vec{H} = \frac{I}{4\pi 5}\left[\frac{5\sqrt{3}}{10} + \frac{5}{5\sqrt{2}}\right](-\vec{a_x}) \times \vec{a_y}$$

$$= \frac{I}{20\pi}\left[\frac{\cancel{5}\overbrace{\sqrt{3}\sqrt{2}}^{\sqrt{6}} + \overset{2}{\cancel{10}}}{\underset{2}{\cancel{10}}\sqrt{2}}\right](-\vec{a_z})$$

$$\boxed{\vec{H} = \frac{-I}{40\sqrt{2}\,\pi}\left[\sqrt{6}+2\right]\vec{a_z} \ [A/m]}$$

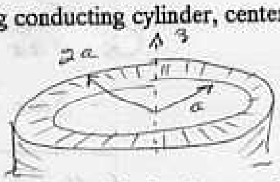
VERSION "A"

$$\boxed{\vec{H} = \frac{+I}{40\sqrt{2}\,\pi}\left[2+\sqrt{6}\right]\vec{a_y} \ [A/m]}$$

Continuation Problem 1:

(b) Magnetization and Bound Volume and Surface Current Densities: [6 marks]

If the magnetic intensity vector field **H** inside a hollow infinitely long conducting cylinder, centered on the z- axis, is given by:



$$H = \frac{3}{\rho}\, a_\phi \;[H/m]$$

The inner radius of the cylinder is "a" [m] and the outer is "2a" [m] and $\mu_r = 3.0$ for the material inside the cylinder.

(i) Develop the expressions for the magnetization vector field **M** [A/m].

$$\therefore \chi_m = \mu_r - 1 = 3 - 1 = 2$$

$$\therefore \boxed{\vec{M} = \chi_m \vec{H} = 2\left(\frac{3}{\rho}\right)\vec{a}_\phi = +\frac{6}{\rho}\,\vec{a}_\phi \;[A/m]}$$

VERSION "A":

$$\boxed{\vec{M} = \frac{9}{\rho}\,\vec{a}_\phi \;[A/m]}$$

(ii)Solve for the volume current density $\mathbf{J}_b$, and the magnetization surface current density $\mathbf{K}_b$ on both surfaces of the cylinder.

$$\vec{J}_b = \nabla \times \vec{M} = \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho\vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & \cancel{\rho}\left(\frac{6}{\cancel{\rho}}\right) & 0 \end{vmatrix} = 0 \qquad \boxed{\therefore \vec{J}_b = 0}$$

$$\underset{@a}{\vec{K}_b} = \vec{M}\times(-\vec{a}_\rho)\Big|_{\rho=a} = \frac{6}{\rho}\,\vec{a}_\phi \times (-\vec{a}_\rho)\Big|_{\rho=a} = +\frac{6}{\rho}\,\vec{a}_z\Big|_{\rho=a}$$

$$\boxed{\therefore \underset{@a}{\vec{K}_b} = +\frac{6}{a}\,\vec{a}_z \;[A/m]} \quad \text{LIKEWISE } \underset{@2a}{\vec{K}_b} = \frac{6}{\rho}\,\vec{a}_\phi\times\vec{a}_\rho\Big|_{\rho=2a}$$

$$\boxed{\therefore \underset{@2a}{\vec{K}_b} = -\frac{3}{a}\,\vec{a}_z \;[A/m]}$$

VERSION "A":

$$\underset{@a}{\vec{K}_b} = +\frac{9}{a}\,\vec{a}_z \;[A/m]$$

$$\underset{@2a}{\vec{K}_b} = -\frac{9}{2a}\,\vec{a}_z \;[A/m]$$

(iii)Determine the expression for the sum of $I_{bv} + I_{bs}$, where "I_{bv}" is the bound volume current and "I_{bs}" is the bound surface current.

$$\therefore \vec{J}_b = 0 \quad \therefore I_{bv} = 0$$

$$\therefore I_{bs} = \oint_{0\to 2\pi} \underset{@2a}{\vec{K}_b} \times \vec{a}_\rho \cdot (2a)\,d\phi\,\vec{a}_\phi + \oint_{2\pi\to 0} \underset{@a}{\vec{K}_b}\times(-\vec{a}_\rho)\cdot a\,d\phi\,\vec{a}_\phi$$

$$= \int_{\phi=0}^{2\pi} -\frac{3}{\cancel{a}}\,\underbrace{\vec{a}_z\times\vec{a}_\rho}_{\vec{a}_\phi}\cdot 2\cancel{a}\,d\phi\,\vec{a}_\phi + \int_{\phi=2\pi}^{0} +\frac{6}{a}\,\underbrace{\vec{a}_z\times(-\vec{a}_\rho)}_{-\vec{a}_\phi}\cdot a\,d\phi\,\vec{a}_\phi$$

$$= -6(2\pi) + \left(-\frac{6}{\cancel{a}}\right)\cancel{a}(-2\pi) = -12\pi + 12\pi = 0$$

$$\therefore \boxed{I_{bv} + I_{bs} = 0}$$

Continuation Problem 1:

(c) Magnetic Boundary Conditions: [6 marks]

VERSION "A": $\vec{B}_1$ = the same

The boundary between two different magnetic materials is given by the equation y = 0.

Region 1: (y < 0) has relative permeability μ_{r1} = 1.0 and $\mathbf{H}_1 = 1\,\mathbf{a}_X + 2\,\mathbf{a}_Y + 3\,\mathbf{a}_Z$ [A/m].

Region 2: (y > 0) has μ_{r2} = 3.0 and a surface current density on the boundary $\mathbf{K} = +4\,\mathbf{a}_Z$ [A/m].

Solve for $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{H}_2$, and $\mathbf{K}_b$ in both regions.

REGION ①:

$\mu_1 = 1.0\mu_0$

$\vec{H}_1 = 1\vec{a_x} + 2\vec{a_y} + 3\vec{a_z}$

$\therefore \boxed{\vec{B}_1 = \mu_0(\vec{a_x} + 2\vec{a_y} + 3\vec{a_z})}$

REGION ②:

$\mu_2 = 3.0\mu_0$

$\vec{a}_{n12} = \vec{a_y}$

$\vec{K} = 4\vec{a_z}$ [A/m]

z, y

But in Region ②

$\chi_m = 3 - 1 = 2$

$\therefore \vec{M} = -6\vec{a_x} + \frac{4}{3}\vec{a_y} + 6\vec{a_z}$

$\therefore \vec{K}_{b2} = \vec{M} \times (-\vec{a_y})$

$= 6\vec{a_z} + 6\vec{a_x}$

$\boxed{\vec{K}_{b2} = 6(\vec{a_x} + \vec{a_z})\ [A/m]}$

$B_{1N} = \vec{B}_1 \cdot \vec{a}_{n12}$

$= \vec{B}_1 \cdot \vec{a_y}$

$= 2\mu_0 = B_{2N}$

$\therefore \vec{B}_{2N} = 2\mu_0 \vec{a_y}$

But $\vec{H}_{2N} = \dfrac{B_{2N}}{3\mu_0} = \dfrac{2}{3}\vec{a_y}$

• Due to $\vec{K}$ there will be components $\vec{H}_{1K}$ & $\vec{H}_{2K}$

• Where $\vec{H}_{1K} = \frac{1}{2}\vec{K} \times \vec{a}_{n12} = \frac{1}{2}(4\vec{a_z}) \times (-\vec{a_y}) = +2\vec{a_x}$ [A/m]

& $\vec{H}_{2K} = \frac{1}{2}(4\vec{a_z}) \times \vec{a_y} = -2\vec{a_x}$ [A/m]

• Therefore $\vec{H}_1 = \vec{H}_1' + \vec{H}_{1K}$ $\therefore \vec{H}_1' = \vec{H}_1 - \vec{H}_{1K}$

$= \vec{a_x} + 2\vec{a_y} + 3\vec{a_z} - 2\vec{a_x}$

$\vec{H}_1' = -\vec{a_x} + 2\vec{a_y} + 3\vec{a_z}$

(NOTE: The (′) FIELD is less the effect of the $\vec{K}$.)

• Therefore $\vec{H}_{1t}' = \vec{H}_1' - \vec{H}_{1n}'$ where $\vec{H}_{1n}' = (\vec{H}_1' \cdot \vec{a}_{n12})\vec{a}_{n12}$

$= (\vec{H}_1' \cdot \vec{a_y})\vec{a_y}$

$= 2\vec{a_y}$

$\therefore \vec{H}_{1t}' = -\vec{a_x} + 2\vec{a_y} + 3\vec{a_z} - 2\vec{a_y}$

$= -\vec{a_x} + 3\vec{a_z}$

But $\vec{H}_{1t}' = \vec{H}_{2t}' = -\vec{a_x} + 3\vec{a_z}$ $\therefore \vec{H}_2' = \vec{H}_{2t} + \vec{H}_{2N}$

$= -\vec{a_x} + 3\vec{a_z} + \frac{2}{3}\vec{a_y}$

$= -\vec{a_x} + \frac{2}{3}\vec{a_y} + 3\vec{a_z}$

• Finally $\vec{H}_2 = \vec{H}_2' + \vec{H}_{2K}$

$= -\vec{a_x} + \frac{2}{3}\vec{a_y} + 3\vec{a_z} - 2\vec{a_x}$

$\boxed{\vec{H}_2 = -3\vec{a_x} + \frac{2}{3}\vec{a_y} + 3\vec{a_z}\ [A/m]}$

$\therefore \vec{B}_2 = 3\mu_0 \vec{H}_2$

$\therefore \boxed{\vec{B}_2 = \mu_0(-9\vec{a_x} + 2\vec{a_y} + 9\vec{a_z})\ [T]}$

In Region ①

$\chi_m = 0$

$\therefore \vec{K}_{b1} = 0$

VERSION "A":

$\vec{H}_2 = -3\vec{a_x} + \vec{a_y} + 3\vec{a_z}$ [A/m] ; $\vec{K}_{b1} = 0$

$\vec{B}_2 = \mu_0(-6\vec{a_x} + 2\vec{a_y} + 6\vec{a_z})$ [T] ; $\vec{K}_{b2} = 3(\vec{a_x} + \vec{a_z})$ [A/m]

Continuation Problem 1:

(d) Induced Currents: [2 marks]

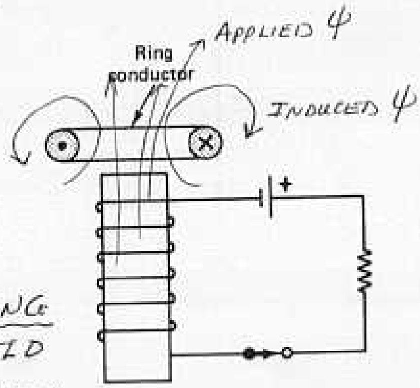
For the diagram shown below circle ***all*** of the correct answers which would cause the current to be flowing in the conducting ring in the direction as shown.

(i) The switch has been closed for a long time.

(ii)The switch has been open for a long time.

(iii)The switch is just being closed.

(iv)The switch is just being opened.



VERSION "A"

(iii) THE SWITCH IS JUST BEING CLOSED

THE INDUCED FIELD IS SUPPORTING THE APPLIED FIELD

∴ THE APPLIED FIELD MUST BE DECREASING.

∴ THE SWITCH IS JUST BEING OPENED!

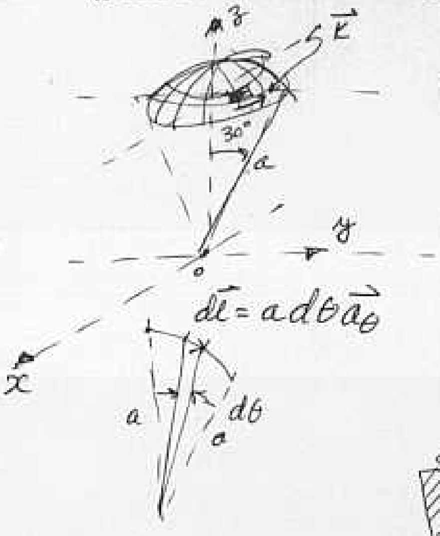
Problem 2: Current, Magnetic Field Vector, Vector Magnetic Potential and Inductance: [20 marks]

(a) A Dome of Surface Current Density: [10 marks]

A dome, defined by $r = a$; $0 \le \theta \le \pi/6$; $0 \le \phi \le 2\pi$, carries a surface current density

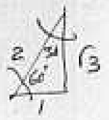
$$\boldsymbol{K} = K_o \sin\theta \ \boldsymbol{a}_\phi \ [A/m]$$

(i) Solve for the total current "I" [A] flowing on the dome. (**Hint:** use $I = \int \boldsymbol{K} \times \boldsymbol{a}_n \cdot d\boldsymbol{l}$)



$$\therefore I = \int_{\theta=0}^{\pi/6} K_o \sin\theta \, \underbrace{\vec{a}_\phi \times \vec{a}_r}_{\vec{a}_\theta} \cdot a\,d\theta\,\vec{a}_\theta$$

$$= aK_o \int_0^{\pi/6} \sin\theta\,d\theta$$



$$= aK_o \left[-\cos\theta\right]_0^{30°}$$

$$= aK_o \left[-\frac{\sqrt{3}}{2} + 1\right]$$

$$\boxed{I = aK_o \left[1 - \frac{\sqrt{3}}{2}\right] [A]}$$

VERSION "A"

$$I = \frac{1}{2} a K_o [A]$$

Continuation Problem 2:

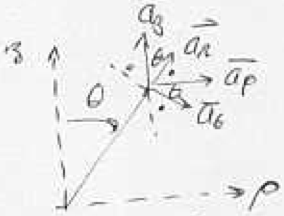
(ii)Derive the magnetic intensity vector field expression **H** at the origin due to the surface current density **K**. Use the incremental version of the ***Biot-Savart Law*** and clearly indicate what are the components **r**, **r'**, **(r - r')**, **|r - r'|**; and what is "ds" in this case.

(Hint: $\int \sin^3\theta\, d\theta = -\int \sin^2\theta\, d\cos\theta = \int (\cos^2\theta - 1)\, d\cos\theta$)

- $d\vec{H} = \dfrac{\vec{K}\, ds \times (\vec{r} - \vec{r}')}{4\pi\, |\vec{r} - \vec{r}'|^3}$; $\vec{r} = 0$; $\vec{r}' = a\vec{a_r}$
  $\therefore\ \vec{r} - \vec{r}' = -a\vec{a_r}$
  $|\vec{r} - \vec{r}'| = a$
- $ds = r^2 \sin\theta\, d\theta\, d\phi$ $\quad |\vec{r} - \vec{r}'|^3 = a^3$
  $= a^2 \sin\theta\, d\theta\, d\phi$
- Therefore $d\vec{H} = \dfrac{K_0 \sin\theta\, \overbrace{\vec{a_\phi} \times (-\cancel{a}\vec{a_r})}^{\vec{a_\theta}}\, \cancel{a}^{2} \sin\theta\, d\theta\, d\phi}{4\pi\, \cancel{a}^{3}}$

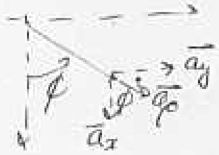
$$d\vec{H} = -\frac{K_0 \sin^2\theta\, d\theta\, d\phi\, \vec{a_\theta}}{4\pi}$$

- But $\vec{H} = \dfrac{-K_0}{4\pi} \displaystyle\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \sin^2\theta\, d\theta\, d\phi\, \vec{a_\theta}$ ... has $\vec{a_\theta} = \vec{a_\theta}(\theta)$!
  $\therefore$ must be converted to rectangular unit vectors.



$\therefore\ \vec{a_\theta} = \cos\theta\, \vec{a_\rho} - \sin\theta\, \vec{a_z}$

- But $\vec{a_\rho} = \cos\phi\, \vec{a_x} + \sin\phi\, \vec{a_y}$

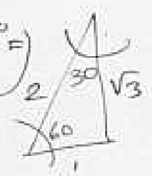


$\therefore\ \vec{a_\theta} = \cos\theta(\cos\phi\, \vec{a_x} + \sin\phi\, \vec{a_y}) - \sin\theta\, \vec{a_z}$

NOTE: That the $\cos\phi$ & $\sin\phi$ terms will become ZERO if $\phi: 0 \to 2\pi$. Therefore the $\vec{a_x}$ and $\vec{a_y}$ term can be dropped. $\therefore\ \vec{a_\theta} = -\sin\theta\, \vec{a_z}$

$$\therefore\ \vec{H_i} = +\frac{K_0}{4\pi} \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \sin^3\theta\, d\theta\, d\phi\, \vec{a_z}$$

$\cos 30° = \dfrac{\sqrt{3}}{2}$



$$= \frac{K_0\, \cancel{2\pi}}{\cancel{4}_{2}\, \cancel{\pi}} \int_{\theta=0}^{\pi/6} (\cos^2\theta - 1)\, d\cos\theta\, \vec{a_z}$$

$$= \frac{K_0}{2}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi/6 = 30°} \vec{a_z}$$

$$= \frac{K_0}{2}\left[\frac{1}{3}\cdot\left(\frac{\sqrt{3}}{2}\right)^3 - \left(\frac{\sqrt{3}}{2}\right) - \frac{1}{3}(1) + 1\right]\vec{a_z} =$$

$$= \frac{K_0}{2}\left[\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{4}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \frac{1}{3} + 1\right]\vec{a_z} \Rightarrow \vec{H} = \frac{K_0}{2}\left[\frac{2}{3} - \frac{3\sqrt{3}}{8}\right]\vec{a_z}\ [A/m]$$

Aside!
$= \dfrac{\sqrt{3}}{8} - \dfrac{4\sqrt{3}}{8} + \dfrac{2}{3}$
$= -\dfrac{3\sqrt{3}}{8} + \dfrac{2}{3}$

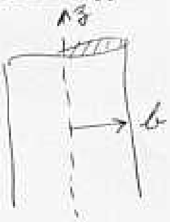
VERSION "A"
$$\boxed{\vec{H} = \frac{5K_0}{48}\vec{a_z}\ [A/m]}$$

Continuation Problem 2:

(b) An Infinitely Long Current Carrying Cylinder centred on the z - axis: [10 marks]

An infinitely long non-magnetic ($\mu = \mu_0$) cylinder of radius "b " [m] centered on the z-axis supports a volume current density given by:

$$J = J_0\,\rho^2\,\mathbf{a}_z\ [A/m]$$



(i) Solve for the total current "I" [A] flowing in the cylinder.

$$I = \int_s \vec{J}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{b} J_0\rho^2\rho\,d\rho\,d\phi$$

$$= \cancel{2}\pi J_0 \frac{\rho^4}{\cancel{4}\,2}\Bigg|_0^b = \frac{b^4\pi J_0}{2}\ [A]$$

$$\therefore\ \boxed{I = \frac{b^4\pi J_0}{2}\ [A]}$$

VERSION "A"

$$\boxed{I = \frac{1}{2}a^4\pi J_0\ [A]}$$

(ii)Derive the expressions for the **H** field inside and outside of the conductor.

VERSION "A" THE SAME

Inside: $(\rho<b)$

$$\oint \vec{H}\cdot d\vec{l} = I_{ENC}$$

$$H_\phi\,2\cancel{\pi}\cancel{\rho} = \frac{\rho^{\cancel{4}\,3}\cancel{\pi}J_0}{2}$$

$$H_\phi = \frac{J_0\rho^3}{4}\quad \therefore\ \boxed{\vec{H} = \frac{J_0\rho^3}{4}\,\vec{a}_\phi\ [A/m]}$$

Outside: $(\rho>b)$

$$\oint \vec{H}\cdot d\vec{l} = I_{ENC}$$

$$H_\phi\,2\cancel{\pi}\rho = \frac{b^4\cancel{\pi}J_0}{2}$$

$$H_\phi = \frac{b^4 J_0}{4\rho}$$

$$\therefore\ \boxed{\vec{H} = \frac{b^4 J_0}{4\rho}\,\vec{a}_\phi\ [A/m]}$$

VERSION "A"

$$\boxed{\vec{H} = \frac{a^4 J_0}{4\rho}\,\vec{a}_\phi\ [A/m]}$$

Continuation Problem 2:

(iii)Solve for the vector magnetic potential **A** inside of the conductor if **A** = 0 at ρ = b [m] using the vector Poisson Equation.

$\vec{\nabla}^2\vec{A} = -\mu_0 \vec{J}$ Since $\vec{J} = J_z \vec{a_z}$ then $\vec{A} = A_z \vec{a_z}$ and due to symmetry considerations $A_z = A_z(\rho)$

$\therefore \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial}{\partial\rho} A_z\right) + \cdots = -\mu_0 J_0 \rho^2$

- multiplying both sides by ρ:

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho} A_z\right) = -\mu_0 J_0 \rho^3$$

- Integrating both sides w.r. to ρ:

$$\rho\frac{\partial}{\partial\rho} A_z = -\frac{\mu_0 J_0 \rho^4}{4} + C_1$$

- Dividing both sides by ρ:

$$\frac{\partial}{\partial\rho} A_z = -\frac{\mu_0 J_0 \rho^3}{4} + \frac{C_1}{\rho}$$

- Integrating both sides w.r.t. ρ for a second time:

$$A_z = -\frac{\mu_0 J_0 \rho^4}{16} + C_1 \ln\rho + C_2$$

- Therefore: $\vec{A} = \left(-\frac{\mu_0 J_0 \rho^4}{16} + C_1 \ln\rho + C_2\right)\vec{a_z}$
- Since $\vec{H} = \frac{J_0 \rho^3}{4}\vec{a_\phi}$ therefore $\vec{B} = \frac{\mu_0 J_0 \rho^3}{4}\vec{a_\phi}$

Then by comparing $\vec{\nabla}\times\vec{A} = \frac{\mu_0 J_0 \rho^3}{4}\vec{a_\phi}$ we find that $C_1 = 0$.

- The constant C_2 can be evaluated from $\vec{A} = 0$ @ $\rho = b$. Therefore:

$$-\frac{\mu_0 J_0 b^4}{16} + C_2 = 0 \quad \text{or} \quad C_2 = +\frac{\mu_0 J_0 b^4}{16}$$

- Therefore: $A_z = -\frac{\mu_0 J_0 \rho^4}{16} + \frac{\mu_0 J_0 b^4}{16}$

$$= \frac{\mu_0 J_0}{16}\left(b^4 - \rho^4\right)$$

- Therefore:

$$\boxed{\vec{A} = \frac{\mu_0 J_0}{16}\left(b^4 - \rho^4\right)\vec{a_z} \, [Wb/m]}$$

VERSION "A"

$$\boxed{\vec{A} = +\frac{\mu_0 J_0}{16}\left(a^4 - \rho^4\right)\vec{a_z} \, [Wb/m]}$$

Continuation Problem 2:

(iv) If the magnetic energy density inside the conductor is given by $w_m = \dfrac{\mu_o J_o^2 \rho^6}{32}$ $[J/m^3]$ calculate the internal self-inductance, L_{int} for an "l" [m] length of the conductor.

$$\therefore W_m = \int_v w_m \, dv = \int_{z=0}^{l}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{b} \frac{\mu_o J_o^2 \rho^6}{32} \rho \, d\rho \, d\phi \, dz$$

$$= \frac{\mu_o J_o^2}{32} \left.\frac{\rho^8}{8}\right|_0^b \left.\phi\right|_0^{2\pi} \left. z\right|_0^l = \frac{\mu_o J_o^2 b^8 2\pi l}{32(8)}$$

$$W_m = \frac{\mu_o b^8 l \pi J_o^2}{128} [J] \quad \text{But}; \; L_{INT} = \frac{2 W_m}{I^2}$$

where $I = \dfrac{b^4 \pi J_o}{2}$ [A]

$$\therefore L_{INT} = \frac{\dfrac{2\mu_o b^8 l \pi J_o^2}{128}}{\dfrac{b^8 \pi^2 J_o^2}{4}} = \frac{2\mu_o l}{128} \cdot \frac{4}{\pi}$$

$$\therefore \boxed{L_{INT} = \frac{\mu_o l}{16\pi} [H]}$$

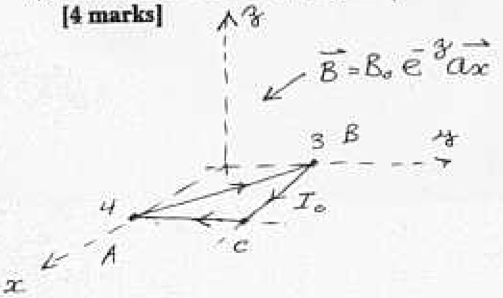
VERSION "A" THE SAME

Problem 3: **Forces, Torques, and Induced EMF: [20 marks]**

A triangular conducting loop connects the points A(4, 0, 0), B(0, 3, 0), and C(4, 3, 0).

(a) A current I_o [A] is flowing in the ABC direction in the circuit in the presence of an external magnetic field $\mathbf{B} = B_o e^{-z} \mathbf{a_x}$ [T].

(i) Determine the force F_{CA} [N] on the current flowing in the CA side of the triangular loop. **[4 marks]**



$$d\vec{F}_{CA} = I_o \, dy \, \vec{a_y} \times B_o e^{-z} \vec{a_x} \quad [z = 0]$$

$$= I_o \, dy \, \vec{a_y} \times B_o \vec{a_x}$$

$$= -I_o B_o \, dy \, \vec{a_z}$$

$$\vec{F}_{CA} = -I_o B_o \int_3^0 dy \, \vec{a_z}$$

$$\therefore \boxed{\vec{F}_{CA} = +3 I_o B_o \vec{a_z} \; [N]}$$

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$$\boxed{\vec{F}_{CA} = +4 I_o B_o \vec{a_z} \; [N]}$$

(ii) Calculate the value of the total force $F_T = F_{AB} + F_{BC} + F_{CA}$ on the circuit. **[2 marks]**

Since $\bar{B}$ is uniform over the three sides of the circuit then

$$\Sigma \vec{F} = 0 \quad \therefore \boxed{\vec{F}_T = 0}$$

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Continuation Problem 3:

(iii)Solve for the torque T_{CA} [Nm] on the side CA taken about the origin.**[5 marks]** $d\vec{T} = \vec{r} \times d\vec{F}$

$$d\vec{T}_{CA} = (4\vec{a}_x + y\vec{a}_y) \times (-I_o B_o \, dy \, \vec{a}_z)$$

$$= -I_o B_o \left(\underbrace{4\vec{a}_x \times \vec{a}_z}_{-\vec{a}_y} + \underbrace{y\vec{a}_y \times \vec{a}_z}_{\vec{a}_x} \right) dy$$

$$= -I_o B_o \left(y\vec{a}_x - 4\vec{a}_y \right) dy$$

$$\vec{T}_{CA} = -I_o B_o \int_{y=3}^{0} (y\vec{a}_x - 4\vec{a}_y)\, dy$$

$$= -I_o B_o \left[\frac{y^2}{2}\vec{a}_x - 4y\vec{a}_y \right]_3^0$$

$$= -I_o B_o \left[-\frac{9}{2}\vec{a}_x + 12\vec{a}_y \right]$$

$$\therefore \boxed{\vec{T}_{CA} = +3 I_o B_o \left(+\frac{3}{2}\vec{a}_x - 4\vec{a}_y \right) [N\text{-}m]}$$

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$$\boxed{\vec{T}_{CA} = 4 I_o B_o \left(2\vec{a}_x - 3\vec{a}_y \right) [N\text{-}m]}$$

(iv)Determine the total torque $T_T = T_{AB} + T_{BC} + T_{CA}$ taken about the point B. **[3 marks]**

$\vec{T}_T = I_o \vec{S} \times \vec{B}$ where $\vec{S} = \frac{1}{2}(3)(4)\vec{a}_z = 6\vec{a}_z$ = Surface area of the cct.

$$\vec{T}_T = I_o \underbrace{6\vec{a}_z \times B_o \vec{a}_x}_{\vec{a}_y} = 6 I_o B_o \vec{a}_y$$

$$\therefore \boxed{\vec{T}_T = 6 I_o B_o \vec{a}_y \; [N\text{-}m]}$$

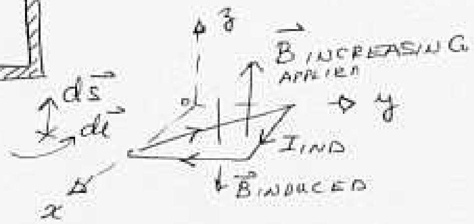
(b) The triangular loop is in the presence of a time-varying magnetic field $\mathbf{B} = B_o \, t \, \mathbf{a}_z$ [T].

(i) Calculate the voltage, V_{emf}, induced in the loop.**[4 marks]**

$$V_{emf} = -\frac{d}{dt}\psi = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = -\frac{d}{dt}\int_S B_o t \vec{a}_z \cdot d\vec{s}$$

$$= -\frac{d}{dt}(B_o t)(6)$$

$$\boxed{V_{emf} = -6 B_o \; [V]}$$



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(ii)Indicate the direction that an induced current would be flowing.**[2 marks]**

$\therefore$ I_{IND} WOULD FLOW IN THE ABC DIRECTION!