* SOLUTIONS *

TO VERSION "A" WITH ANSWERS FOR VERSION "B"

Ryerson University

Department of Electrical and Computer Engineering

ELE401: Field Theory

Term Test #1

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Sections: 01 - 08
Time: ~ 1 .75 hrs

Instructions:

- **Closed book test.** Aids such as vector operators and integral tables will be provided if needed. **No questions are to be asked during the test.** If some aspect of a question is unclear make suitable assumptions, i.e. write down the assumptions, and continue solving the problem. **Calculators will not be required and are not permitted.**
- **Answer all three (3) questions.** Weight of each question is indicated. Part marks will be given. Remember to **simplify** all answers and indicate their **units,** and clearly indicate by **underlining** or **"boxing"** the final answers.
- Show all work in the space provided. **If a *"Right"* answer appears without any indication on how it was derived that answer will be considered to be *"Wrong"*.** Use the back sides of pages if more working space is needed. ***Note:*** **(60) marks** are available; **(40) marks** is taken as a full paper, therefore the extra **(20) marks** are **bonus marks**.

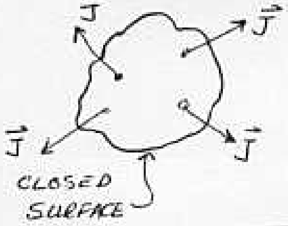
Questions	*Available Marks*	*Marks Achieved*
Question 1	20 marks	
Question 2	20 marks	
Question 3	20 marks	
Total Marks (60) *Full Marks (40)*	*60 marks available* *(40 marks required)*	

Name (Please Print)	
Student Number	
Circle Your Section Number	01, 02, 03, 04, 05, 06, 07, 08

Problem 1: **Three Short Questions [20 marks total]**

(a) Derivations: [6 marks]

(i) Show how the "Continuity of Current" Equation $\nabla \cdot \mathbf{J} = - \partial \rho_v / \partial t$ can be derived from the concept that the total current "*I*" flowing out from a closed surface is equal to the rate of decrease of the charges contained in the closed volume. [2] MARKS TOTAL.



CURRENT OUT $I = \oint_S \vec{J} \cdot d\vec{s} = -\frac{d}{dt} Q_{ENC}$

Divergence Theorem

$$= -\frac{d}{dt}\int_v \rho_v \, dv$$

$$= -\int_v \frac{\partial}{\partial t}\rho_v \, dv$$

$$\int_v \underbrace{\vec{\nabla}\cdot\vec{J}}\, dv = +\int_v \underbrace{-\frac{\partial \rho_v}{\partial t}}\, dv$$

EQUATING THE INTEGRANDS

$$\therefore \boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\rho_v}$$

(ii)Using the Continuity of Current Equation as the starting point derive the so-called "defusion of charge" equation: [4] MARKS TOTAL

$$\rho_v(t) = \rho_v(0)\, e^{-(t/T_r)} \quad \text{where } T_r = \varepsilon / \sigma$$

$\therefore \vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\rho_v$, BUT $\vec{J} = \sigma\vec{E}$ & $\vec{E} = \frac{\vec{D}}{\epsilon}$

$\therefore \vec{\nabla}\cdot\frac{\sigma}{\epsilon}\vec{D} = -\frac{\partial}{\partial t}\rho_v$ OR $\frac{\sigma}{\epsilon}\vec{\nabla}\cdot\vec{D} = -\frac{\partial}{\partial t}\rho_v$

SINCE σ, ϵ ARE NOT DEPENDENT ON SPACE VARIABLES.

BUT $\vec{\nabla}\cdot\vec{D} = \rho_v$ $\therefore$ THE EQUATON BECOMES:

$$\frac{\sigma}{\epsilon}\rho_v = -\frac{\partial}{\partial t}\rho_v$$

$\therefore$ DIVIDING BOTH SIDES BY ρ_v & MULTIPLYING BOTH SIDES BY ∂t & REARRANGING YIELDS.

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\epsilon}\partial t \xrightarrow[\text{AS...}]{\text{SAME}} \frac{d\rho_v}{\rho_v} = -\frac{\sigma}{\epsilon}dt$$

NEXT INTEGRATING BOTH SIDES...

$$\int_{\rho_v(0)}^{\rho_v(t)} \frac{d\rho_v}{\rho_v} = -\frac{\sigma}{\epsilon}\int_{t=0}^{t} dt \Rightarrow \ln \rho_v \Big|_{\rho_v(0)}^{\rho_v(t)} = -\frac{\sigma}{\epsilon}t \Rightarrow \ln\left(\frac{\rho_v(t)}{\rho_v(0)}\right) = -\frac{\sigma}{\epsilon}t$$

FINALLY TAKING EXPONETIALS OF BOTH SIDES YIELDS

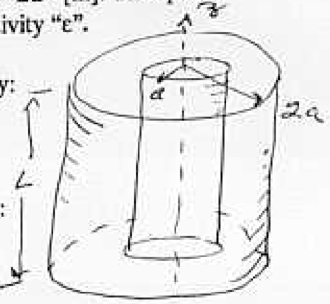
$$\frac{\rho_v(t)}{\rho(0)} = e^{-\frac{\sigma}{\epsilon}t} \xrightarrow[\text{QED.}]{\therefore} \therefore \underbrace{\rho_v(t) = \rho_v(0)\, e^{-\frac{\sigma}{\epsilon}t} = \rho_v(0)\, e^{-\frac{t}{T_r}}}_{\text{MARKS}}$$

VERSION "B" SAME AS "A".

Continuation: 1. Three Short Questions:

(b) Fields , Voltages, and Energy Inside A Coaxial Conductor. [6 marks]

A coaxial infinitely long conductor centred on the z-axis consists of a solid conducting cylinder with radius ***"a"*** [m], and a thin cylindrical conduction outer shell with a radius ***"2a"*** [m]. The space between the conducting surfaces is filled with a dielectric material with permittivity "ε".



(i) If the voltage at any point between the conducting surfaces is given by:

$$V(\rho) = +(3\,V_o / \ln(2))\,\ln(2\,a/\rho)\ \ [V]$$

and the surface charge density on the inner conductor is found to be:

$$\rho_s = +3\,\varepsilon\,V_o / (a\,\ln(2))\ \ [C/m^2]$$

Determine the energy stored in an *"L"* [m] length of the coaxial conductor using

$$W_E = ½ \int_s \rho_s V_s\, ds\ \ [J] \qquad [2]\ \text{MARK TOTAL}$$

$$\therefore\ V(\rho) = +\frac{3V_o}{\ln(2)}\cdot\ln\left(\frac{2a}{\rho}\right)\ [V]$$

$$\therefore\ V(a) = +\frac{3V_o}{\cancel{\ln(2)}}\cdot\cancel{\ln}\left(\frac{2\cancel{a}}{\cancel{a}}\right) = +3V_o\ [V] = V_s$$

$$\therefore\ W_E = \frac{1}{2}\int_S \rho_s V_s\, ds$$

$$= \frac{1}{2}\int_S \frac{3\epsilon V_o}{a\ln(2)}(3V_o)\,ds = \frac{1}{2}\cdot\frac{9\epsilon V_o^2}{a\ln(2)}\int_S ds$$

$$= \frac{1}{\cancel{2}}\,\frac{9\epsilon V_o^2}{\cancel{a}\ln(2)}\cdot(\cancel{2}\pi\cancel{a}L) \qquad \text{SINCE } \int_S ds = 2\pi a L_1$$

$$\boxed{W_E = \frac{9\pi\epsilon V_o^2 L}{\ln(2)}\ [J]}$$

VERSION "B":

$$W_E = \frac{4\epsilon\pi V_o^2 L}{\ln(3)}\ [J]$$

Cont.: **1. Three Short Questions:**

Cont.: **(b) Fields, Voltage and Energy Inside A Coaxial Conductor.**

(ii)If instead of knowing the voltage, the the electric field E is known at any point between the conducting surfaces and is given by:

$$\mathbf{E}(\rho) = +(3\,V_o / (\ln(2)\,\rho))\ \mathbf{a}_\rho \quad [\mathrm{V/m}]$$

determine the energy stored in an "*L*" [m] length of the conductor using the relationship:

$$W_E = \tfrac{1}{2} \int_v \mathbf{D} \cdot \mathbf{E}\, dv \quad [\mathrm{J}]$$

where the field exist only in the space between the conducting surfaces. [4] MARKS TOTAL

$$\because\ \vec{E} = \frac{3V_o}{\ln(2)\rho}\,\vec{a_\rho}\ [\mathrm{V/m}] \quad \therefore\ \vec{D} = \frac{3\epsilon V_o}{\ln(2)\rho}\,\vec{a_\rho}\ [\mathrm{C/m^2}]$$

$$\therefore\ W_E = \frac{1}{2}\int_v \vec{D}\cdot\vec{E}\,dv = \frac{1}{2}\int_v \frac{3\epsilon V_o}{\ln(2)\rho}\cdot\frac{3V_o}{\ln(2)\rho}\,dv$$

$$= \frac{1}{2}\cdot\frac{9\epsilon V_o^2}{(\ln(2))^2}\int_{z=0}^{L}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{2a}\frac{1}{\rho^{\cancel{2}}}\,\cancel{\rho}\,d\rho\,d\phi\,dz$$

[1/2] MARKS

$$= \frac{1}{\cancel{2}}\,\frac{9\epsilon V_o^2}{(\ln(2))^{\cancel{2}}}\,[L]\,[\cancel{2}\pi]\,\underbrace{[\ln\rho]_a^{2a}}_{\ln\left(\frac{2a}{a}\right) = \cancel{\ln(2)}}$$

$$\boxed{W_E = \frac{9\pi\epsilon V_o^2 L}{\ln(2)}\ [\mathrm{J}]}$$

VERSION "B":

$$W_E = \frac{4\epsilon\pi V_o^2 L}{\ln(3)}\ [\mathrm{J}]$$

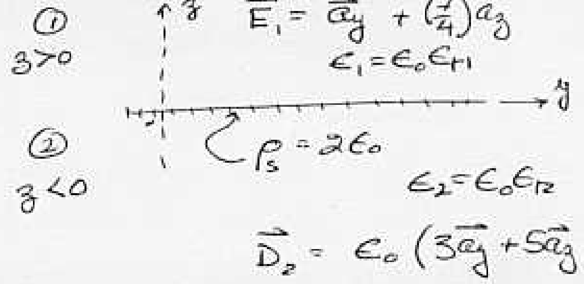
***Cont.:* 1. Three Short Questions:**

(c) Boundary Conditions.[8 marks]

Two different dielectric material meet on the z = 0 plane. ***Region #1*** is considered to be z > 0 ; and ***Region #2*** is z < 0. The expressions for the electric field intensity vectors $\mathbf{E}_1$ and the electric flux density vector field $\mathbf{D}_2$ in their respective regions are given by:

$$\mathbf{E}_1 = 1.0\,\mathbf{a}_Y + (7/4)\,\mathbf{a}_Z \ \text{[V/m]} ; \text{ and } \quad \mathbf{D}_2 = \varepsilon_o\,(3\,\mathbf{a}_Y + 5\,\mathbf{a}_Z) \ [\text{C/m}^2]$$

Determine the values of the two relative permittivities ε_{R1} and ε_{R2}, and solve for the electric field $\mathbf{E}_2$ and the electric flux density field $\mathbf{D}_1$ if the surface charge density on the boundary between the two dielectrics is $\rho_S = 2\,\varepsilon_o$ [C/m²].



$\because\ \vec{E}_{1t} = \vec{E}_{2t}$

$\therefore\ \vec{E}_{2t} = 1.0\,\vec{a}_y$

BUT $\vec{D}_{2t} = \epsilon_o \epsilon_{r2} \vec{E}_{2t}$

$= \epsilon_o \epsilon_{r2}(1.0\,\vec{a}_y) = \epsilon_o 3\,\vec{a}_y \quad \therefore \boxed{\epsilon_{r2} = 3}$ [2 MARKS]

$\because\ D_{1N} - D_{2N} = \rho_s \quad \therefore\ \epsilon_o \epsilon_{r1} E_{1N} - D_{2N} = \rho_s$

OR $\cancel{\epsilon_o}\epsilon_{r1}\left(\frac{7}{4}\right) - \cancel{\epsilon_o}\,5 = 2\cancel{\epsilon_o}$

$\therefore\ \epsilon_{r1}\left(\frac{7}{4}\right) = 7 \quad \therefore \boxed{\epsilon_{r1} = 4}$ [2 MARKS]

$\therefore\ \vec{D}_1 = \epsilon_o \epsilon_{r1} \vec{E}_1 = \epsilon_o 4\left(1.0\,\vec{a}_y + \frac{7}{4}\vec{a}_z\right)$

$\vec{D}_1 = \epsilon_o(4\vec{a}_y + 7\vec{a}_z)$ [C/m²] [2 MARKS]

LIKEWISE: $\vec{E}_2 = \dfrac{\vec{D}_2}{\epsilon_o \epsilon_{r2}} = \dfrac{1}{\cancel{\epsilon_o}\epsilon_{r2}}\cancel{\epsilon_o}(3\vec{a}_y + 5\vec{a}_z)$

$= \frac{1}{3}(3\vec{a}_y + 5\vec{a}_z)$

$\vec{E}_2 = \vec{a}_y + \frac{5}{3}\vec{a}_z$ [V/m] [2 MARKS]

[IF ϵ_{r1} & ϵ_{r2} ARE WRONG THE MISTAKES MAY HAVE TO BE "CARRIED" INTO THE SOLUTIONS FOR $\vec{D}_1$ & $\vec{E}_2$]

[[8]] MARKS TOTAL.

• VERSION "B" SAME AS "A".

Problem 2. **A Non-Uniform Surface Charge Distribution. [20 marks total]**

In free space a non-uniform surface charge density $\rho_S = \alpha\rho$ [C/m²] (where α is a constant) is found to exist on the surface defined by:

$$z = 0;\ a \le \rho \le b;\ \pi/2 \le \phi \le \pi$$

(a) Determine the total charge "*Q*" on the surface. **[4 marks]**

$$Q = \int_S \rho_s\, ds = \int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=a}^{b} (\alpha\rho)\,\rho\, d\rho\, d\phi = \alpha \int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=a}^{b} \rho^2 d\rho\, d\phi$$

$$= \alpha \frac{\rho^3}{3}\Big|_a^b \phi\Big|_{\pi/2}^{\pi} = \frac{\alpha}{3}\left[b^3 - a^3\right]\left[\frac{\pi}{2}\right]$$

$$\boxed{Q = \frac{\alpha\pi}{6}(b^3 - a^3)\ [C]}$$

VERSION "B:"

$$Q = \frac{b\pi}{6}(\beta^3 - \alpha^3)\ [C]$$

(b) Develop an expression for the voltage "*V*" at the origin, assuming that the voltage is zero at infinity. **[6 marks]**

$$V(0) = \int_S \frac{\rho_s}{4\pi\epsilon_0 R}\, ds \ ; \quad R = |\vec{r} - \vec{r}\,'|$$

$$\vec{r} = 0;\quad \vec{r}\,' = +\rho\,\vec{a}_\rho$$

$$|\vec{r} - \vec{r}\,'| = |-\rho\,\vec{a}_\rho| = \rho$$

$$= \frac{1}{4\pi\epsilon_0}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=a}^{b} \frac{\alpha\rho}{\cancel{\rho}}\,\cancel{\rho}\, d\rho\, d\phi$$

$$= \frac{\alpha\cancel{\pi}}{4\cancel{\pi}\epsilon_0 2}\ \frac{\rho^2}{2}\Big|_a^b = \frac{\alpha}{8\epsilon_0}(b^2 - a^2)\ [V]$$

$$\boxed{V(0) = \frac{\alpha}{16\epsilon_0}(b^2 - a^2)\ [V]}$$

VERSION "B:"

$$V(0) = \frac{b}{16\epsilon_0}(\beta^2 - \alpha^2)\ [V]$$

Cont.: 2. A Non-Uniform Charge Distribution.

(c) Calculate the electric field **E** at the origin due to the surface charge density . Express the final answer in three formats: (i) in Cartesian unit vectors, (ii) in Cylindrical unit vectors and (iii) in Spherical unit vectors. **[10 marks]**

$$d\vec{E} = \frac{dQ\,(\vec{r}-\vec{r}\,')}{4\pi\epsilon_0 |\vec{r}-\vec{r}\,'|^3} \quad \text{WHERE} \quad dQ = \rho_s\,dS = (\alpha\rho)(\rho\,d\rho\,d\phi) = \alpha\rho^2 d\rho\, d\phi$$

$$d\vec{E} = \frac{\alpha \cancel{\rho^2}\, d\rho\, d\phi\, (-\rho\,\vec{a}_\rho)}{4\pi\epsilon_0\, \cancel{(\rho)^3}}$$

$\vec{r} = 0$

$\vec{r}\,' = \rho'\vec{a}_\rho$

$\vec{r}-\vec{r}\,' = -\rho'\vec{a}_\rho$

$|\vec{r}-\vec{r}\,'| = \rho'$

SINCE $\vec{r}=0$ THE OTHER VARIABLES DO NOT NEED TO BE PRIMED

$$= \frac{-\alpha}{4\pi\epsilon_0}\, d\rho\, d\phi\, \vec{a}_\rho$$

$$\therefore\ \vec{E}(0) = -\frac{\alpha}{4\pi\epsilon_0}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=a}^{b} d\rho\, d\phi\, \vec{a}_\rho$$

$$= -\frac{\alpha}{4\pi\epsilon_0}\, \rho\Big|_a^b \int_{\phi=\frac{\pi}{2}}^{\pi} d\phi\, \vec{a}_\rho$$

BUT $\vec{a}_\rho = \cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y$

$$= -\frac{\alpha(b-a)}{4\pi\epsilon_0}\int_{\phi=\pi/2}^{\pi} \vec{a}_\rho\, d\phi$$

$$= \frac{-\alpha(b-a)}{4\pi\epsilon_0}\int_{\phi=\pi/2}^{\pi} (\cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y)\, d\phi$$

$$= \frac{-\alpha(b-a)}{4\pi\epsilon_0}\left[\left(\overset{0}{\sin\pi} - \overset{1}{\sin\tfrac{\pi}{2}}\right)\vec{a}_x + \left(-\overset{(-1)}{\cos\pi} + \overset{0}{\cos\tfrac{\pi}{2}}\right)\vec{a}_y\right]$$

$$\vec{E}(0) = \frac{-\alpha(b-a)}{4\pi\epsilon_0}\left[-\vec{a}_x + \vec{a}_y\right] = +\frac{\alpha(b-a)}{4\pi\epsilon_0}\left[\vec{a}_x - \vec{a}_y\right]$$

VERSION "B": $\vec{E} = +\frac{b(\beta-\alpha)}{4\pi\epsilon_0}\left[\vec{a}_x - \vec{a}_y\right]$ [V/m]

$\vec{E} = +\frac{b(\beta-\alpha)}{4\pi\epsilon_0}\left[\vec{a}_\rho - \vec{a}_\phi\right]$ [V/m]

$\vec{E} = +\frac{b(\beta-\alpha)}{4\pi\epsilon_0}\left[\vec{a}_\theta - \vec{a}_\phi\right]$ [V/m]

$$\&\ \vec{E}(0) = \frac{+\alpha(b-a)}{4\pi\epsilon_0}\left[\vec{a}_\rho - \vec{a}_\phi\right]\ [\text{V/m}]$$

$$\&\ E(0) = \frac{+\alpha(b-a)}{4\pi\epsilon_0}\left[\vec{a}_\theta - \vec{a}_\phi\right]\ [\text{V/m}]$$

Problem 3: **Concentric Spherical Shells. [20 marks total]**

A spherical structure consists of two concentric conducting shells centred at the origin. The inner conducting shell (#1) has a radius = ***"a"*** [m] while the outer shell (#2) has a radius = ***"2a"*** [m].

The space between the conducting surfaces is filled at various times with different dielectric and/or conducting materials.

(a) Linear, Homogeneous, Isotropic Dielectric Material. [6 marks]

When the dielectric material between the conducting shells is homogeneous it is found to have a relative dielectric constant $\varepsilon_R = 3.0$. The electric flux density vector **D**(r) becomes :

$$\mathbf{D}(r) = +(\alpha\, a^2 / r^2)\ \mathbf{a}_r\ \ [\mathrm{C/m^2}]\ ;\ \text{where } \alpha \text{ is a constant.}$$

(i) Determine the surface charge densities ρ_{S1} and ρ_{S2} on the inner and outer conductor surfaces respectively.

[2] marks

$$\rho_{S1} = \vec{D}\cdot\vec{a}_r \Big|_{r=a} = \frac{\alpha a^2}{r^2}\Big|_{r=a} = \alpha\ [\mathrm{C/m^2}]$$

$$\rho_{S2} = \vec{D}\cdot(-\vec{a}_r)\Big|_{r=2a} = -\frac{\alpha a^2}{r^2}\Big|_{r=2a} = -\frac{\alpha \cancel{a^2}}{4\cancel{a^2}} = -\frac{\alpha}{4}\ [\mathrm{C/m^2}]$$

VERSION "B":

$$\rho_{S1} = \beta\ [\mathrm{C/m^2}]$$

$$\rho_{S2} = \frac{-\beta}{4}\ [\mathrm{C/m^2}]$$

(ii)Determine the total charge ***"Q₂"*** on the surface of the outer (#2) conductor.

[1] mark

$$Q_2 = \int_S \rho_{S2}\, ds = +\int_S -\frac{\alpha}{4}\, ds = -\frac{\alpha}{4}\int_S ds = -\frac{\alpha}{4}\left(4\pi r^2\right)\Big|_{r=2a}$$

$$Q_2 = -\frac{\alpha}{\cancel{4}}\cdot\left(4\pi\, \cancel{4} a^2\right) = -4\alpha\pi a^2\ [\mathrm{C}]$$

VERSION "B":

$$Q_2 = -4\pi b^2 \beta\ [\mathrm{C}]$$

(iii)If the voltage at any point inside the capacitor is given by the expression:

[2] marks

$$V(r) = \alpha\, a\,(2a - r)/(6\,\varepsilon_0\, r)\ \ [\mathrm{V}]$$

determine the voltage between the two conducting surfaces, i.e. find ***"V_{12}"***, the voltage on surface #1 with respect to surface #2.

$$V(a) = \frac{\alpha \cancel{a}\,(2a - a)}{6\,\varepsilon_0\, \cancel{a}} = \frac{\alpha a}{6\varepsilon_0}\ [\mathrm{V}]$$

$$V(2a) = \frac{\alpha a\,(2a - 2a)}{6\varepsilon_0\,(2a)} = 0\ [\mathrm{V}]$$

$$\therefore\ V_{12} = +\frac{\alpha a}{6\varepsilon_0}\ [\mathrm{V}]$$

VERSION "B":

$$V_{12} = +\frac{\beta b}{6\varepsilon_0}\ [\mathrm{V}]$$

VERSION "B:"

$C = 24\epsilon_0 \pi b$ [F]

Cont.: 3. Concentric Spherical Shells.

Cont.: (a) Linear, Homogeneous, Isotropic Dielectric Material.

(iv) Solve for "*C*" the capacitance of the concentric spherical shell structure.

[1] MARKS

$$C = \frac{Q}{V} = \frac{4\cancel{\chi}\pi a^{\cancel{2}}}{\frac{\cancel{\chi}\cancel{a}}{6\epsilon_0}} = 24\epsilon_0\pi a \text{ [F]}$$

[[6]] MARKS TOTAL FOR (a).

* IN THIS QUESTION ANY ERRORS IN "Q" OR "V" SHOULD BE "CARRIED"

(b) Conducting Material. [6 marks]

The dielectric material used in part (a) is replace with a weakly conducting material with permittivity $\varepsilon = \varepsilon_0$ and conductivity σ [S/m]. The voltages on the conducting shells are now changed so that the outer shell (#2) is at $-2V_o$ [V] and the inner one is at $+3V_o$ [V]; where V_o is a constant.

[3] MARKS (i) Use Laplace's Equation, $\nabla^2 V = 0$, to solve for V(r), the expression which gives the voltage at any point in space between the two conducting surfaces. Evaluate the constants of integration from the known boundary values.

$$\vec{\nabla}^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \text{ or } \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0, \text{ IF } r \neq 0$$

INTEGRATING ONCE:

$$r^2\frac{\partial V}{\partial r} = C_1 \; ; \; \therefore \frac{\partial V}{\partial r} = \frac{C_1}{r^2}$$

INTEGRATING ONCE MORE:

$$\boxed{V(r) = -\frac{C_1}{r} + C_2}$$

NEXT EVALUATING C_1 & C_2:

$$@\, r = a;\; V(a) = +3V_o = -\frac{C_1}{a} + C_2 \quad (1)$$

$$\&\; @\, r = 2a;\; V(2a) = -2V_o = \frac{-C_1}{2a} + C_2 \rightarrow (2)$$

SUBTRACT EQN (2) FROM EQN (1)

$$+5V_o = \frac{-C_1}{a} + \frac{C_1}{2a} = C_1\left(\frac{1}{2a} - \frac{1}{a}\right) = \frac{-C_1}{2a}$$

$$\therefore C_1 = -10aV_o$$

$\therefore$ SOLVING FOR C_2 NEXT:

$$+3V_o = -\frac{(-10\cancel{a}V_o)}{\cancel{a}} + C_2 \quad \therefore C_2 = -7V_o$$

$$\therefore \boxed{V(r) = +\frac{10aV_o}{r} - 7V_o \quad [V]}$$

VERSION "B": $V(r) = +\frac{10bV_o}{r} - 7V_o$ [V]

Cont.: 3. Concentric Spherical Shells.

Cont.: (b) Conducting Material.

[2] (ii)If the electric field intensity vector E(r) inside the spherical structure is given by the expression:

$$E(r) = +(10\,a\,V_o / r^2)\ \mathbf{a}_r\ [V/m]$$

determine the current density vector J(r) and from it solve for the total current "*I*" flowing between the conducting surfaces.

$$\therefore\ \vec{J} = \frac{10a\sigma V_o}{r^2}\vec{a_r}\ [A/m^2]$$

$$\therefore\ I = \int_s \vec{J}\cdot d\vec{s}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{10a\sigma V_o}{\cancel{r^2}}\vec{a_r}\cdot \cancel{r^2}\sin\theta\, d\theta\, d\phi\, \vec{a_r}$$

$$= 10a\sigma V_o\,[2\pi]\,\underbrace{\left[-\cos\theta\right]_0^{\pi}}_{-\cos\pi+\cos 0^\circ = +2}$$

$$I = 40a\sigma\pi V_o\ [A]$$

VERSION "B:"

$$I = 40\sigma\pi b V_o\ [A]$$

[1] (iii)Determine the expression for the resistance "*R*" between the two conducting surfaces.

$$R = \frac{V}{I} = \frac{\overset{1}{\cancel{5V_o}}}{\underset{8}{\cancel{40}}a\sigma\pi \cancel{V_o}}$$

$$\boxed{R = \frac{1}{8a\sigma\pi}\ [\Omega]}$$

VERSION "B:"

$$R = \frac{1}{8\sigma\pi b}\ [\Omega]$$

* AGAIN ANY ERRORS IN "V" AND/OR "I" MUST BE "CARRIED".

[6] MARKS TOTAL IN (b)

Cont.: 3. Concentric Spherical Shells.

(c) A Linear, Non-Homogeneous, Isotropic, Non-Conducting Dielectric.[8 marks]

The dielectric material between the conducting spheres is changed to a linear, non-conduction, non-homogeneous, isotropic material. The relative permittivity, ε_R, is now a function of space variables, specifically $\varepsilon_R = \varepsilon_R(r)$. The expression for polarization vector **P** [C/m²] inside the dielectric is found to be given by :

$$\mathbf{P}(r) = +(\alpha a^2/r^2)((6a-3r)/(7a-3r))\ \mathbf{a}_r\ [\mathrm{C/m^2}]$$

while the electric flux density field **D**(r) is the same as it was in part (a) i.e.

$$\mathbf{D}(r) = +(\alpha a^2/r^2)\ \mathbf{a}_r\ [\mathrm{C/m^2}]$$

[2] MARKS (i) Determine the (bound) polarization surface charge densities ρ_{PS1} and ρ_{PS2} on the dielectric material's inner and outer surfaces respectively.

$$\rho_{PS1} = \vec{P}\cdot(-\vec{a}_r)\Big|_{r=a} = -\frac{\alpha a^2}{r^2}\frac{6a-3r}{7a-3r}\Big|_{r=a}$$

$$\rho_{PS1} = -\frac{\alpha a^2}{a^2}\frac{6a-3a}{7a-3a} = -\alpha\frac{3a}{4a} = -\frac{3}{4}\alpha\ [\mathrm{C/m^2}]$$

$$\rho_{PS2} = \vec{P}\cdot(+\vec{a}_r)\Big|_{r=2a} = -\frac{\alpha a^2}{r^2}\frac{6a-3r}{7a-3r}\Big|_{r=2a} = 0$$

VERSION "B": $\rho_{PS1} = -\frac{3}{4}\beta\ [\mathrm{C/m^2}]$

[3] MARKS (ii)Evaluate the (bound) polarization volume charge density ρ_{PV} inside the dielectric material.

$$\rho_{PV} = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 P_r\right\}$$

[1/2] MARKS

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left\{\cancel{r^2}\frac{\alpha a^2}{\cancel{r^2}}\frac{6a-3r}{7a-3r}\right\}$$

$$= -\frac{\alpha a^2}{r^2}\left\{\frac{-3}{7a-3r} - \frac{-3(6a-3r)}{(7a-3r)^2}\right\}$$

$$= -\frac{\alpha a^2}{r^2}\left\{\frac{-3(7a-3r)+3(6a-3r)}{(7a-3r)^2}\right\}$$

$$= -\frac{\alpha a^2}{r^2}\left\{\frac{-21a\ \cancel{+9r}+18a\ \cancel{-9r}}{(7a-3r)^2}\right\}$$

$$= -\frac{\alpha a^2}{r^2}\frac{-3a}{(7a-3r)^2}$$

$$\therefore \rho_{PV} = \frac{+3\alpha a^3}{r^2(7a-3r)^2}\ [\mathrm{C/m^3}]$$

VERSION "B": $\rho_{PV} = \frac{3\beta b^3}{r^2(7b-3r)^2}\ [\mathrm{C/m^3}]$

Cont.: 3. Concentric Spherical Shells.

Cont.: (c) A Linear, Non-Homogeneous, Isotropic, Non-Conducting Dielectric.

[3] MARKS (iii) Since the material is linear, although non-homogeneous, the the basic relationships still apply; i.e.

$$\mathbf{D} = \varepsilon_O \mathbf{E} + \mathbf{P} \;;\; \mathbf{P} = \varepsilon_O X_e \mathbf{E} \;;\text{and } \varepsilon_R = 1 + X_e$$

Use the given **P** and **D** fields to solve for the expression for $\varepsilon_R(r)$.

$$\because X_e = \epsilon_r - 1 \quad \therefore \vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E}$$

$$= \cancel{\epsilon_0}(\epsilon_r - 1)\frac{\vec{D}}{\cancel{\epsilon_0}\epsilon_r}$$

$$= \left(\frac{\epsilon_r - 1}{\epsilon_r}\right)\vec{D}$$

$$\therefore \frac{\cancel{\alpha a^2}(6a - 3r)}{\cancel{r^2}(7a - 3r)} = \frac{\epsilon_r - 1}{\epsilon_r} \cdot \frac{\cancel{\alpha a^2}}{\cancel{r^2}}$$

$$\therefore (7a - 3r)(\epsilon_r - 1) = (6a - 3r)\epsilon_r$$

$$\text{OR} \quad [(7a - 3r) - (6a - 3r)]\epsilon_r = 7a - 3r$$

$$(7a - \cancel{3r} - 6a + \cancel{3r})\epsilon_r = 7a - 3r$$

$$a\,\epsilon_r = 7a - 3r$$

$$\therefore \boxed{\epsilon_r = 7 - \frac{3r}{a}}$$

[8] MARKS TOTAL FOR (C)

VERSION "B":

$$\epsilon_r = 7 - \frac{3r}{b}$$