Chapter 2 - The Time Value of Money

1 Reasons for Interest

Lender: Interest compensates for administrative costs of lending, risk of default, and the opportunity cost of money.
Borrower: Interest is the premium paid to avoid waiting for money.

Implicit here is the "earning power of money." For money to earn something, both the lender and borrower must wait. There is time between the receipt and the return: the *time value of money*. In Engineering Economics, money can be viewed as the productivity expected from the resource "money."

2 Simple Interest

- directly proportional to the principal (capital) involved in the loan.

$$I = PiN$$

where:
I = interest income
P =principal
i = interest rate per period
N = number of periods

Example: loan of \$1000 at 10% per year for 2 years (P=\$1000, i=0.10, N=2).
Interest over the period of the loan is

$$I = (\$1000)(0.10)(2) = 200$$

The future value of the loan (F) is principal plus interest, or

$$F = P + I = P + PiN = P(1 + iN) = 1000 + 200 = 1200$$

What if N is not a full year?
(a) Ordinary Simple Interest: 1 YEAR = 360 DAYS (or 1 MONTH = 30 DAYS)
(b) Exact Simple Interest: 1 YEAR = 365 DAYS (so N=fraction of year that loan is in effect)
Example: loan of \$1000 at 10% per year for 2 months (January and February).
Ordinary Simple Interest:

$$F = P(1 + iN) = 1000\left(1 + \frac{2}{12}\right) = 1000(1.01667) = \$1016.67$$

Exact Simple Interest:

$$F = P(1 + iN) = 1000\left(1 + \frac{31 + 28}{365}\right) = 1000(1.01616) = \$1016.16$$

3 Compound Interest

- includes charges for accumulated interest as well as unpaid principal (interest on interest).

Example: Loan of \$1000 at 10% for 2 years

Year	Beginning Amount	Interest	Amount Owed
1	\$1000	1000(0.10)=100	\$1100
2	\$1100	1100(0.10)=110	\$1210

The future value of this 2 year loan can be written

$$\begin{aligned} F_2 &= (P + Pi) + (P + Pi)i \\ &= (P + Pi)(1 + i) \\ &= P(1 + i)^2 \\ &= 1000(1 + 0.10)^2 = \$1210 \end{aligned}$$

In general,

$$F_N = P(1 + i)^N$$

4 Nominal Interest Rates (r)

Interest rates are normally quoted on an annual basis.

Example: "8 percent compounded quarterly" or "8 percent compounded daily"

More frequent compounding increases the interest paid on a loan (or, equivalently, increases the future value of the loan, F).

5 Effective Interest Rates (i)

- rate actually paid on the loan.

$$i = \left[\frac{F - P}{P}\right] 100$$

Example: suppose $F = 1196$, $P = 1000$. Then

$$i = \left[\frac{1196 - 1000}{1000}\right] 100 = 19.6\%$$

Without using future and present values we can find the effective rate by

$$i = \left(1 + \frac{r}{m}\right)^m - 1$$

where i is the effective rate of interest, r is the nominal rate, and m is the number of compounding periods per year.

Example: what is the effective rate of interest for an nominal interest rate of 18% compounded semi-annually (twice per year)?

$$i = \left(1 + \frac{0.18}{2}\right)^2 - 1 = 0.1881 = 18.81\%$$

This implies that 18.81% compounded *annually* and 18% compounded *semi-annually* are equivalent.

6 Continuous Compounding

- $m \to \infty$

$$i = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m - 1 = e^r - 1$$

where $e = 2.71828$...Why?

$$\left(1 + \frac{r}{m}\right)^m = \left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^r$$

and

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Example: $r = 18.232\%$

$$i = e^{0.18232} - 1 = 0.20 = 20\%$$

Example: $i = 22.1\%$

$$\begin{aligned} e^r - 1 &= 0.221 \\ e^r &= 1.221 \\ r \ln e &= \ln(1.221) \\ r &= 0.20 = 20\% \end{aligned}$$

7 Time Value Equivalence

- relate time and earnings to find amounts which are equivalent over time.

Examples

1) $P = 1000$, $i = 0.10$, $N = 2$

$$F = 1000(1+0.10)^2 = \$1210$$

or \$1000 today is equivalent to \$1210 in two years.

2) $F = 1000$ two years from now is equivalent to \$826.44 today if $i = 10\%$:

$$P = \frac{F}{(1+i)^2} = \$826.44$$

3) If $i = 10\%$, \$868 is equivalent to \$500 received in one year plus \$500 received in two years:

$$P = \frac{500}{(1+0.10)} + \frac{500}{(1+0.10)^2} = 455 + 413 = \$868$$

8 Conversion Factors and Symbols

(a) Conversion factors of a single amount to a future or present value.
(b) Annuity factors: equal payments (A), equal periods between payments (N), and first payment at the *end* of first period. An annuity is a uniform series of values.

Factor Name	Find	Given	Symbol
Compound Factor	F	P	$(F/P, i, N)$
Present Value	P	F	$(P/F, i, N)$
Sinking Fund	A	F	$(A/F, i, N)$
Series Compound	F	A	$(F/A, i, N)$
Capital Recovery	A	P	$(A/P, i, N)$
Series Present Value	P	A	$(P/A, i, N)$
Arithmetic Gradient	A	G	$(A/G, i, N)$

$G =$ given uniform increase in annuity amount

9 Relations

$$(F/P, i, N) = \frac{1}{(P/F, i, N)}$$

$$(A/F, i, N) = \frac{1}{(F/A, i, N)}$$

$$(A/P, i, N) = \frac{1}{(P/A, i, N)}$$

$$(F/A, i, N) = (F/P, i, N)(P/A, i, N)$$
$$(F/P, i, N) = (F/A, i, N)(A/P, i, N)$$

$$(A/P, i, N) = (A/F, i, N) + i$$

10 Formula Development

- Compound Amount Factor - find F given P.

$$F_1 = P + Pi = P(1+i)$$
$$F_2 = P + Pi + (P + Pi)i = P(1+i)^2$$

and so for any N,

$$F_N = P(1+i)^N = P(F/P, i, N)$$

$(F/P, i, N)$ is given in Appendix D of the text for different values of i and N.
Example: $(F/P, 10, 3) = (1 + 0.10)^3 = 1.331$.

- Present Value Factor - find P given F.

$$P = F\frac{1}{(1+i)^N} = F(P/F, i, N)$$

- Sinking Fund Factor - find A given F.

$$A = \frac{Fi}{[(1+i)^N - 1]} = F(A/F, i, N)$$

$(A/F, i, N)$ is given in Appendix D of the text for different values of i and N.
The first payment (made at the end of period 1) earns interest for the remaining $N-1$ periods, or becomes $A(1+i)^{N-1}$. Treating each subsequent payment in the same way,

$$F = A\left[(1+i)^{N-1} + (1+i)^{N-2} + ... + 1\right] \qquad ((1))$$

Multiplying this by $(1+i)$ yields

$$(1+i)F = A\left[(1+i)^N + (1+i)^{N-1} + ... + (1+i)\right] \qquad ((2))$$

Thus, from (2)-(1),

$$\begin{aligned} F(1+i) - F &= A(1+i)^N - A \\ Fi &= A\left[(1+i)^N - 1\right] \\ A &= \frac{Fi}{[(1+i)^N - 1]} \end{aligned}$$

Example: Given $F = \$5866$, $i = 8\%$, $N = 5$,

$$A = \frac{5866(0.08)}{[(1.08)^5 - 1]} = \$1000$$

Therefore, to accumulate \$5866 in 5 years when $i = 8\%$, equal amounts of \$1000 must be paid at the end of each of the five years.

- **Series Compound Amount Factor** - find F given A.

$$F = \frac{A\left[(1+i)^N - 1\right]}{i} = A(F/A, i, N)$$

$(F/A, i, N)$ is given in Appendix D of the text for different values of i and N.

Example: Given $A = \$1000$, $i = 8\%$, $N = 5$,

$$F = 1000(F/A, 8, 5) = 1000(5.8666) = \$5866.6$$

- **Capital Recovery Method** - find A given P (or, find the annuity payment amount which will completely dissipate the given present value, given i and N).

$$A = P\frac{i(1+i)^N}{[(1+i)^N - 1]} = P(A/P, i, N)$$

$(A/P, i, N)$ is given in Appendix D of the text for different values of i and N.
The present value of an annuity A over N periods is

$$P = \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + ... + \frac{A}{(1+i)^N} \qquad ((1))$$

Multiplying this by $\frac{1}{1+i}$ yields

$$\frac{P}{(1+i)} = \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + ... + \frac{A}{(1+i)^{N+1}} \qquad ((2))$$

From (2)-(1),

$$P\left[\frac{1}{(1+i)} - 1\right] = A\left[\frac{1}{(1+i)^{N+1}} - \frac{1}{(1+i)}\right]$$

Multiplying by $(1+i)$

$$P\left[1 - (1+i)\right] = A\left[\frac{1}{(1+i)^N} - 1\right]$$

so that

$$A = P\frac{i}{\left[\frac{1}{(1+i)^N} - 1\right]} = P\frac{i(1+i)^N}{[(1+i)^N - 1]}$$

Example: $P = \$3993$, $i = 8\%$, $N = 5$.

$$A = 3993(A/P, 8, 5) = 3993(0.25046) = \$1000$$

It would take five equal payments of \$1000 to repay a debt of \$3993.

It is straightforward to show that $(A/P, i, N) = (A/F, i, N) + i$,

$$\begin{aligned}
\frac{i(1+i)^N}{[(1+i)^N - 1]} &= \frac{i}{[(1+i)^N - 1]} + i \\
&= \frac{i}{[(1+i)^N - 1]} + \frac{i\left[(1+i)^N - 1\right]}{[(1+i)^N - 1]} \\
&= \frac{i(1+i)^N}{[(1+i)^N - 1]}
\end{aligned}$$

- Series Present-Value Factor - find P given A.

Solving the capital recovery formula for P,

$$P = A\left[\frac{\left[(1+i)^N - 1\right]}{i(1+i)^N}\right] = A(P/A, i, N)$$

$(P/A, i, N)$ is given in Appendix D of the text for different values of i and N.

Example: $A = \$1000$, $i = 8\%$, $N = 5$.

$$P = 1000(P/A, 8, 5) = 1000(3.9926) = \$3993$$

This implies that \$3993 today is equivalent to receiving \$1000 per year for each of the next five years, if the interest rate is 8%.

- Arithmetic-Gradient Conversion Factor - converts a non-uniform (growing) annuity into an equivalent uniform one. The annuity grows every period by a constant amount G, starting from a base A' paid at the end of the first N periods. The earnings profile of this annuity is then

$$A', A' + G, A' + 2G, A' + 3G, ..., A' + (N-1)G$$

This converts to a uniform payment A^* according to

$$A^* = A' + G(A/G, i, N)$$

where

$$(A/G, i, N) = \frac{1}{i} - \frac{N}{i}(A/F, i, N)$$

again available in Appendix D.
Example: If $A^{'} = 500$, $G = 300$, $i = 8\%$, $N = 5$,

$$\begin{aligned}
A^* &= A' + G(A/G, i, N) \\
&= 500 + 300(A/G, 8, 5) \\
&= 500 + 300(1.8463) \\
&= 1053.89
\end{aligned}$$

This implies that 5 equal end-of-period payments of \$1053.89 are equivalent to five payments starting at \$500 and increasing by \$300 each period. Keep in mind that G can be negative (if the payments decrease each period).

- Geometric Series Conversion Factor - useful in calculating present worth when a series grows at a constant percentage rate, g, per period (g can again be negative), starting from a base value $A^{'}$. For a given N and i, the present worth depends on the relation between i and g.

Case 1: $g > i$

$$P = \frac{A'}{1+i}(F/A, i^*, N)$$

where

$$i^* = \frac{1+g}{1+i} - 1$$

Example: $g = 10\%$, $i = 8\%$.

$$i^* = \frac{1+0.10}{1+0.08} - 1 = 1.85\%$$

So that, if $A^{'} = 1000$ and $N = 5$,

$$P = \frac{1000}{1.08}(F/A, 1.85, 5) = 926(5.1882) = \$4804$$

Here it is necessary to interpolate in the table in Appendix D to find $(F/A, 1.85, 5)$.

Case 2: $g = i$

$$P = \frac{NA'}{1+i}$$

Case 3: $g < i$

$$P = \frac{A'}{1+g}(P/A, i^*, N)$$

where

$$i^* = \frac{1+g}{1+i} - 1$$

11 Discounted Cash Flow Calculations

Convert cash flow problems into equivalent sums that are useful in comparing investments.

- **Cash Flow Diagrams** - useful in showing the structure of a problem.

Example: \$2000 loan paid off in 3 installments at $i = 10\%$

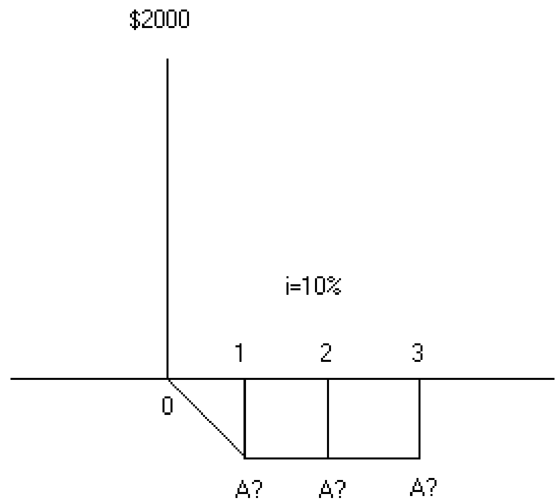

Figure 1: Example cash flow diagram

12 Calculation of Time Value Equivalents: Single Payment Cash Flow

Find i or N that makes F and P equivalent.
<u>Unknown i</u>: At what i will \$1000 be invested today to be worth \$2000 in 9 years ($N = 9$, $F = 2000, P = 1000$)?

$$\begin{aligned} F &= P(F/P, i, N) \\ \frac{F}{P} &= (F/P, i, N) \\ \frac{2000}{1000} &= (F/P, i, 9) \end{aligned}$$

The solution can be found in the Tables in appendix D ($i = 8\%$).
<u>Unknown N</u>: If $i = 8\%$ per annum, how long will it take for \$1000 invested today to become \$2000?

$$\begin{aligned} \frac{F}{P} &= (F/P, i, N) = (1+i)^N \\ 2 &= (1.08)^N \\ \ln(2) &= N \ln(1.08) \\ N &= \frac{\ln(2)}{\ln(1.08)} = 9 \end{aligned}$$

13 Calculation of Time Value Equivalents: Multiple Payment Cash Flows

- More Compounding Periods than Payments

Example: Deposit \$500 in 2000, 2002 and 2004. How large will your account be in 2006 if $i = 7\%$?

$$\begin{aligned} F &= 500(F/P,7,6) + 500(F/P,7,4) + 500(F/P,7,2) \\ &= 500(1.5007) + 500(1.3107) + 500(1.1449) \\ &= \$1978 \end{aligned}$$

- Annuity with Unknown i

Example: A machine is purchased for \$8065 will reduce costs by \$2020 per year, and will last 5 years. What is the rate of return on the investment?

$$\begin{aligned} P &= A(P/A,i,N) \\ \frac{P}{A} &= (P/A,i,5) \\ 3.993 &= (P/A,i,5) \end{aligned}$$

Therefore $i = 8\%$ from the Tables.

- Annuity Due - annuity due at the *beginning* instead of the end of a period.

Example: Find P for a series of 15 year-end payments of \$1000 each where the first payment is due today and $i = 5\%$.

$$\begin{aligned} P &= A + A(P/A,5,14) \\ &= 1000 + 1000(9.8995) \\ &= \$10899 \end{aligned}$$

- Deferred Annuity - annuity with first payment made *later* than the end of the first period.

Example: At $i = 6\%$, what is the value today (year 2000) of a series of payments of \$317.70 starting at the end of 2006 and continuing for 4 more years (2007, 2008, 2009, 2010)? The present value of the annuity at the end of 2005 is

$$P(2005) = A(P/A,6,5) = 317.70(4.2123) = \$1338.25$$

This becomes the future value in finding the present value today:

$$\begin{aligned} P(2000) &= 1338.25(P/F,6,5) \\ &= 1338.25(0.74726) \\ &= 1000 \end{aligned}$$

- Present Value of an Arithmetic Gradient

Example: Sign a lease at $20000 per year with annual increase of $1500 per year for 8 years. Payments start at the end of this year. If $i = 7\%$, what amount paid today would be equivalent to the 8-year lease payment plan? Here, $A' = 20000$, $G = 1500$, $i = 7\%$.

$$A = A' + G(A/G, 7, 8) = 20000 + 1500(3.1463) = \$24719.45$$

Therefore

$$P = A(P/A, 7, 8) = 24719.45(5.9712) = \$147,604.78$$

- Income and Outlay - cash flow involves both receipts and disbursements.

Example: A new production method involves labour savings of $7000 per year over the 5 year life of the project. Immediate costs are $12000, and annual maintenance costs would be $4000. If $i = 6\%$, should the firm implement the project?

Assuming costs and savings occur at the end of the year, *net* annual savings are

$$A = \$7000 - 4000 = \$3000$$

For the project to be at all worthwhile, the net return (in present value) must be greater than the $12000 initial cost.

$$P = 3000(P/A, 6, 5) = 3000(4.21236) = \$12,637$$

Another way to approach this problem is from capital recovery. The required net savings over the 5 years to meet the obligation of $12000 is

$$A = P(A/P, 6, 5) = 12,000(0.23740) = \$2849$$

which is less than the actual savings of $3000. These two methods are identical (the $637 gain in the first method is the present value of the $151 savings for each of the 5 years)

14 Calculations with Continuous Interest

With discrete compounding:

$$F = P(1+i)^N$$

where

$$i = \left(1 + \frac{r}{m}\right)^m - 1$$

With continuous compounding:

$$F = Pe^{rN}$$

e^{rN} corresponds to $(1+i)^N$, and $i = e^r - 1$ as well.

15 Continuous Compounding, Discrete Cash Flow

Future value:

$$F = Pe^{rN}$$

Present value

$$P = Fe^{-rN}$$

Sinking fund

$$A = F\left[\frac{e^r - 1}{e^{rN} - 1}\right]$$

Series compound factor

$$F = A\left[\frac{e^{rN} - 1}{e^r - 1}\right]$$

Capital recovery

$$A = P\left[\frac{e^{rN}(e^r - 1)}{e^{rN} - 1}\right]$$

Series present worth factor

$$P = A\left[\frac{e^{rN} - 1}{e^{rN}(e^r - 1)}\right]$$

Note: when the same *effective* interest rate is used, discrete and continuous compounding give the same results (see text).

16 Continuous Compounding, Continuous Cash Flow

Each period's payment is received continuously in small amounts over the period. The total amount accumulated over the year is $\bar{A}$ and r is the continuous interest rate per period.

$$F = \bar{A}\int_{t=0}^{N} e^{rt}dt = A\left(\frac{e^{rN} - 1}{r}\right)$$

It is easy to derive the other formulas as well (see Table 2.5 in the text)

Notation	Formula
$(F/P, r, N)$	$F = P\left[\frac{e^{rN}-1}{re^r}\right]$
$(P/F, r, N)$	$F = P\left[\frac{e^r-1}{re^{rN}}\right]$
$(\bar{A}/F, r, N)$	$\bar{A} = F\left[\frac{r}{e^{rN}-1}\right]$
$(F/\bar{A}, r, N)$	$F = \bar{A}\left[\frac{e^{rN}-1}{r}\right]$
$(\bar{A}/P, r, N)$	$\bar{A} = P\left[\frac{re^{rN}}{e^{rN}-1}\right]$
$(P/\bar{A}, r, N)$	$P = \bar{A}\left[\frac{e^{rN}-1}{re^{rN}}\right]$