There are generally several ways to solve any given problem. In any question for which you did not get full marks, study the sketch solutions below and be sure you understand them and can supply intermediate steps.

Part A. Multiple-Choice [3 marks each] (Choose one answer in each case.)

1. The integral $\int_2^6 \frac{dx}{x-3}$:

   (A) converges to ln(3). (B) converges to ln(3/2). (C) converges to ln(2/3).

   (D) converges to ln(6). (E) diverges. (Integral improper)

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2. The integral $\int x\, e^{2x}\, dx$ may be evaluated to give:

   (A) $(x+1)e^{2x}+C$ (B) $(2x+1)e^{2x}+C$ (C) $(x-1)e^{2x}+C$

   (D) $\frac{(x-1)}{2}e^{2x}+C$ (E) $\frac{(x-1/2)}{2}e^{2x}+C$. (Integration by parts)

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3. Integration by parts is based on a differentiation rule. That rule is:

   (A) The chain rule. (B) The sum rule. (C) The product rule.

   (D) The quotient rule. (E) The power rule.

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4. Consider the following statements.

   i) If a function f is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x)=0$ then $\int_0^\infty f(x)dx$ converges.

   ii) If a function f is continuous on $[0,\infty)$ and $\int_0^\infty f(x)dx$ diverges then $\lim_{x\to\infty} f(x)\neq 0$.

   iii) If f' is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x)=0$ then $\int_0^\infty f'(x)dx=-f(0)$.

   Choose the correct option below.

   A) Only i) is correct. B) Only ii) is correct. C) Only iii) is correct.

   D) i), ii), and iii) are all correct. E) Only i) and iii) are correct.

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5. Let $I=\int \cos^2(x)\, dx$ then:

   A) $I=\frac{x}{2}+\frac{1}{4}\sin(2x)+const.$ B) $I=\frac{1}{2}(x+\sin(x)\cos(x))+const.$

   C) $I=\frac{x}{2}+\frac{1}{4}(\sin(x)+\cos(x))^2+const.$ D) All of the above. E) None of the above.

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6. Consider the following limits:

   i) $\lim_{x\to 0+}\frac{\cos(x)}{x}$ ii) $\lim_{x\to 0+}(1+x)^{1/x}$ iii) $\lim_{x\to 0+}(\tan x)^x$

   Which of the above is an indeterminate form?

   A) Only i). B) Only ii). C) Only iii). D) Only i) and ii). E) Only ii) and iii).

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7. Consider the following integrals:

   i) $\int_0^\infty \sin(1/x)\, dx$ ii) $\int_0^4 \frac{x}{x^2-2}dx$ iii) $\int_0^\pi \sec(x)dx$ iv) $\int_0^2 \frac{x^3-2x^2+x}{x-1}\, dx$

   Of these integrals, which one(s) is(are) improper?

   A) All. B) Only i). C) Only i) and iv). D) Only i), ii) and iii). E) Only i), ii) and iv).

Part B. Full-Answer

8. [3 marks] Write out the partial fraction expansion of the following rational function. **Do not evaluate the coefficients**. $f(x)=\frac{x^2(x-2)+x(x-2)^2}{x^2(x-2)^3(x^2+1)^2}=\frac{x+(x-2)}{x(x-2)^2(x^2+1)^2}$ so

$$f(x)=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}$$

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9. [12 marks] Evaluate the following integrals.

(a) $I = \int \frac{\sqrt{x^2-9}}{x^2}\,dx \qquad x > 3.$ Let $x = 3\sec\theta \quad dx = 3\sec\theta\tan\theta\ d\theta$

Then $I = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int (\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C$

x $\quad$ $\sqrt{x^2-a^2}$

θ

a

From the triangle we see that this gives

$$I = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x} + C$$

(b) $I = \frac{1}{2}\int \frac{5x^{1/2}-14}{x^{3/2}-5x+4x^{1/2}}dx.$ Let $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$ and

$I = \int \frac{5u-14}{u^2-5u+4}\,du = \int \frac{5u-14}{(u-1)(u-4)}\,du$

Partial fraction expansion of this gives:

$\frac{5u-14}{(u-1)(u-4)} = \frac{3}{u-1} + \frac{2}{u-4}$

so $I = 3\ln|u-1| + 2\ln|u-4| + C$ or, substituting back,

$I = 3\ln|\sqrt{x}-1| + 2\ln|\sqrt{x}-4| + C$

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10. [7 marks] Calculate $y = \lim_{x\to 0^+}(\sin x)^x$.

$$\begin{aligned}
\ln y &= \lim_{x\to 0^+} x\ln(\sin x) \qquad [0.\infty] \\
&= \lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}} \qquad [\tfrac{\infty}{\infty}] \\
&\overset{l'H}{=} \lim_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{\sin x} \qquad [\tfrac{0}{0}] \\
&\overset{l'H}{=} \lim_{x\to 0^+} \frac{-2x}{\cos x} = 0
\end{aligned}$$

Thus $y = e^{(\ln y)} = e^0 = 1$

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11. [7 marks] Determine the convergence/divergence of the following integral and evaluate if convergent.

$I_0 = \int_1^\infty \frac{\ln x}{x^2}\,dx.$ Let $I = \int \frac{\ln x}{x^2}\,dx$. Let $u = \ln x$ so $dv = \frac{dx}{x^2}$ and $du = \frac{dx}{x}$ with $v = \frac{-1}{x}$. Integration by parts then gives:

$$\begin{aligned}
I &= \frac{-\ln x}{x} + \int \frac{dx}{x^2} \\
&= -\frac{1+\ln x}{x} + C
\end{aligned}$$

The improper integral is then:

$$\begin{aligned}
I_0 &= \lim_{t\to\infty}\left(-\frac{1+\ln x}{x}\right)\Big|_1^t \\
&= 1 - \lim_{t\to\infty}\left(\frac{1+\ln t}{t}\right) \qquad [\tfrac{\infty}{\infty}] \\
&\overset{l'H}{=} 1 - \lim_{t\to\infty}\left(\frac{\frac{1}{t}}{1}\right) \\
&= 1
\end{aligned}$$